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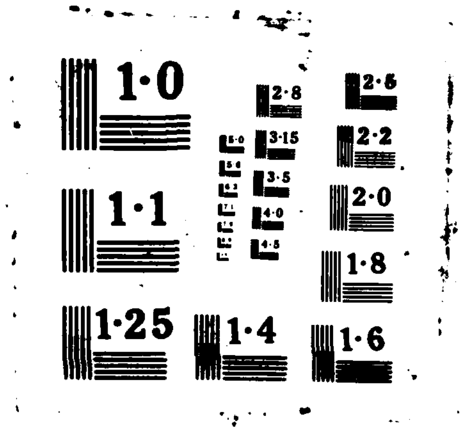
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ESTIMATE FOR THE ERRORS IN EIGENVALUE AND EIGENVECTOR APPROXIMATION BY GALERKIN METHODS, WITH PARTICULAR ATTENTION TO THE CASE OF MULTIPLE EIGENVALUES

by

I. Babuška and J. E. Osborn

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Refined estimates for the errors in eigenvalue and eigenvector approximation by finite element, or, more generally, Galerkin methods, as they apply to self-adjoint problems, are presented. Particular attention is given to the case of multiple eigenvalues. The results are new in this case. The proof is based on a novel approach which yields the known results for simple eigenvalues in a simple way. Numerical computations are presented and analyzed in light of the theoretical results.

Estimates for the Errors in Eigenvalue and Eigenvector Approximation by Galerkin Methods, with Particular Attention to the Case of Multiple Eigenvalues

by

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Dedicated to W.C. Rheinboldt on his 60th birthday

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Abstract

Refined estimates for the errors in eigenvalue and eigenvector approximation by finite element, or, more generally, Galerkin methods, as they apply to self-adjoint problems, are presented. Particular attention is given to the case of multiple eigenvalues. The results are new in this case. The proof is based on a novel approach which yields the known results for simple eigenvalues in a simple way. Numerical computations are presented and analyzed in light of the theoretical results.

1. Introduction.

It is the purpose of this paper to derive some refined estimates for the errors in eigenvalue and eigenvector approximation by finite element, or, more generally, Galerkin methods, as they apply to self-adjoint problems. The results are new in the case of multiple eigenvalues. The proof is based on a novel approach which yields the known results for simple eigenvalues in a simple way.

Suppose λ_k is an eigenvalue of multiplicity q of a self-adjoint problem and let $M(\lambda_k)$ denote the space of eigenvectors corresponding to λ_k, where $\|\cdot\|$ denotes the energy norm for our problem. Let S be the finite dimensional approximation space employed in the Galerkin method. λ_k will be approximated from above by q of the Galerkin approximate eigenvalue:

$$\lambda_k \le \lambda_{S,k}, \lambda_{S,k+1}, \ldots, \lambda_{S,k+q-1}.$$

If we choose the $\lambda_{S,i}$ in increasing order then we have

$$\lambda_k \le \lambda_{S,k} \le \cdots \le \lambda_{S,k+q-1}.$$

Our main estimate for the error in eigenvalue approximation is

$$\lambda_{S,k} - \lambda_k \le C \left[\inf_{\substack{u \in M(\lambda_k) \\ \|u\|=1}} \inf_{\chi \in S} \|u-\chi\| \right]^2 = C \varepsilon_{\lambda_k}(S)^2, \tag{1.1}$$

thus showing that the error between λ_k and $\lambda_{S,k}$, the approximate eigenvalue closest to λ_k, is, to within a multiplicative constant, the square of the minimal energy norm distance between exact eigenvectors $u \in M(\lambda_k)$ with $\|u\| = 1$ and S, i.e., the square of the energy norm distance between S and the eigenvector

$u \in M(\lambda_k)$ with $\|u\| = 1$ that can be best approximated by S. For $\lambda_{S,k+q-1} - \lambda_k$, the error between λ_k and $\lambda_{S,k+q-1}$, the approximate eigenvalue farthest from λ_k, we prove

$$(1.2) \qquad \lambda_{S,k+q-1} - \lambda_k \le C\left[\sup_{\substack{u \in M(\lambda_k) \\ \|u\|=1}} \inf_{\chi \in S} \|u-\chi\|\right]^2 = C\bar{\varepsilon}_{\lambda_k}(S)^2,$$

and for the errors $\lambda_{S,k+i} - \lambda_k, i = 1,\ldots,q-2$, we obtain bounds in terms of quantities intermediate in size between $C\varepsilon_{\lambda_k}(S)^2$ and $C\bar{\varepsilon}_{\lambda_k}(S)^2$.

These results should be contrasted with those in the literature. In Babuška and Aziz [1], Fix [4], and Kolata [6], the estimates

$$(1.3) \qquad \lambda_{S,k+i} - \lambda_k \le C\bar{\varepsilon}_{\lambda_k}(S),\ i = 0,\ldots,q-1,$$

are proved. For $i = 0,\ldots,q-2$, (1.3) is weaker than the estimates stated above ((1.1) for $i = 0$ and those mentioned after (1.2) for $i = i,\ldots,q-2$); for $i = q-1$, (1.3) is the same as (1.2) In Birkhoff, de Boor, Swartz, and Wendroff [2], which presents the earliest results of the general type we are discussing, the eigenvalue estimates depend on the sum of the squares of the energy norm distances between S and the unit eigenvectors associated with all the eigenvalues λ_j not exceeding λ_k. The feature of (1.1) that is new is the dependence of $\varepsilon_{\lambda_k}(S)$ on only one eigenvector $u \in M(\lambda_k)$, namely the one best approximated by S.

Regarding the errors in the approximate eigenvectors, we show that if $u_{S,k}$ is the Galerkin approximate eigenvector corresponding to $\lambda_{S,k}$, then there is a $u_k = u_k(S) \in M(\lambda_k)$ with $\|u_k\| = 1$

such that

$$(1.3)\qquad \|u_{S,k} - u_k\| \le C\varepsilon_{\lambda_k}(S).$$

The error $\|u_{S,k+q-1} - u_{k+q-1}\|$ is bounded by $C\bar{\varepsilon}_{\lambda_k}(S)$ and the errors $\|u_{S,k+i} - u_{k+i}\|$, $i = 1,\ldots,q-2$, are bounded by quantities intermediate in size between $\varepsilon_{\lambda_k}(S)$ and $\bar{\varepsilon}_{\lambda_k}(S)$. The best previously known result is

$$(1.4)\qquad \|u_{S,k+i} - u_{k+i}\| \le C\bar{\varepsilon}_{\lambda_k}(S),\ i = 0,\ldots,q-1.$$

In Section 2 we introduce the class of variationally formulated, self-adjoint eigenvalue problems considered in the paper, define the Galerkin approximations to these problems, and in Lemmas 2.1 - 2.3 give the preliminary results which are used in the sequel. The main theoretical result of the paper is presented and proved in Section 3. The treatment is direct and self-contained, relying on a minimal amount of functional analysis background. In Section 4 we present numerical computations for a finite element approximation of a problem with double eigenvalues for which each double eigenvalue has associated eigenvectors of strikingly different approximation properties. The quantities

$$\sup_{\substack{u\in M(\lambda_k)\\ \|u\|=1}} \inf_{\chi\in S} \|u-\chi\|^2$$

and

$$\inf_{\substack{u\in M(\lambda_k)\\ \|u\|=1}} \sup_{\chi\in S} \|u-\chi\|^2$$

are thus of different sizes and we would therefore expect $\lambda_{S,k}-\lambda_k$ and $\lambda_{S,k+1}-\lambda_k$ to be of different sizes. This is clearly shown by

the computations. The computations also show that $u_{S,k}$ (the approximate eigenvector belonging to the approximate eigenvalue closest to λ_k) converges to an exact eigenvector with good approximation properties, while $u_{S,k+1}$, the approximate eigenvector belonging to the approximate eigenvalue farthest from λ_k, converges to an exact eigenvector with poor approximation properties.

The literature on eigenvalue problems is extensive, with many papers bearing, at least tangentially, on the problem addressed in this paper. We have, however, mentioned only those papers that bear directly on the central theme of our results; namely, the Galerkin approximation of eigenpairs corresponding to multiple eigenvalues. For a general treatment of eigenvalue problems and their literature, we refer to the excellent and comprehensive monograph of Chatelin [3].

2. Preliminaries.

Suppose H is a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, respectively, and suppose we are given two symmetric bilinear forms $B_0(u,v)$ and $D(u,v)$ on $H\times H$. $B_0(u,v)$ is assumed to satisfy

$$(2.1) \qquad |B_0(u,v)| \le C_1\|u\|\|v\|, \ \forall\ u,v \in H$$

and

$$(2.2) \qquad C_2\ \|u\|^2 \le B_0(u,u), \ \forall\ u \in H, \quad \text{with} \quad C_2 > 0.$$

It follows from (2.1) and (2.2) that $(u,v)_{B_0} = B_0(u,v)$ and $\|u\|_{B_0} = (B_0(u,u))^{1/2}$ are equivalent to (u,v) and $\|u\|$, respectively. Regarding D, we suppose

$$(2.3) \qquad 0 < D(u,u), \ \forall\ 0 \ne u \in H,$$

and that

$$(2.4) \qquad \|u\|_D = (D(u,u))^{1/2}$$

is compact with respect to $\|\cdot\|$, i.e., it has the property that from any subsequence which is bounded in $\|\cdot\|$, one can extract a subsequence which is Cauchy in $\|\cdot\|_D$.

We then consider the variationally formulated, self-adjoint eigenvalue problem

$$(2.5) \qquad \begin{cases} \text{Seek}\ \lambda\ (\text{real}) \ \text{and}\ 0 \ne u \in H \ \text{such that} \\ B_0(u,v) = \lambda D(u,v), \ \forall\ v \in H. \end{cases}$$

Under the assumptions we have made, there is a sequence of eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots \nearrow +\infty$$

and corresponding eigenvectors

$$u_1, u_2, \ldots$$

which can be chosen to satisfy

$$B_0(u_i, u_j) = \lambda_i D(u_i, u_j) = \delta_{i,j}, \tag{2.6}$$

where $\delta_{i,j} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. Furthermore, any $u \in H$ can be written as

$$u = \sum_{j=1}^{\infty} a_j u_j, \quad \text{with} \quad a_j = B_0(u, u_j), \tag{2.7}$$

where (2.7) converges in $\|\cdot\|_{B_0}$. The eigenvalues λ_j satisfy the following well-known variational principles:

$$\begin{aligned} \lambda_1 &= \min_{u \in H} \frac{B_0(u,u)}{D(u,u)} = \frac{B(u_1,u_1)}{D(u_1,u_1)}, \\ \lambda_k &= \min_{\substack{u \in H \\ B_0(u,u_i)=0, \\ i=1,2,\ldots,k-1}} \frac{B_0(u,u)}{D(u,u)} = \frac{B(u_k,u_k)}{D(u_k,u_k)}, \quad k = 2,3,\ldots \end{aligned} \tag{2.8}$$

(the minimum principle)

and

$$\lambda_k = \min_{\substack{V_k \subset H \\ \dim V_k = k}} \max_{u \in V_k} \frac{B_0(u,u)}{D(u,u)} = \max_{u \in U_k = sp(u_1,\ldots,u_k)} \frac{B_0(u,u)}{D(u,u)}, \quad k = 1,2,\ldots \tag{2.9}$$

(the minimum-maximum principle).

For any λ_k we let

(2.10) $M(\lambda_k) = \{u : u$ is an eigenvector of (2.5) corresponding to $\lambda_k\}$.

We shall be interested in approximating the eigenpairs of (2.5) by finite element, or, more generally, Galerkin methods.

Toward this end we suppose we are given a (one parametric) family $\{S_h\}_{0<h\le 1}$ of subspaces $S_h \subset H$ and we consider the eigenvalue problem

$$(2.11)\qquad \begin{cases} \text{Seek } \lambda_h \text{ (real)}, \ 0 \neq u_h \in S_h \text{ such that} \\ B_0(u_h,v) = \lambda_h D(u_h,v), \ \forall\, v \in S_h. \end{cases}$$

The eigenpairs (λ_h,u_h) of (2.11) are then viewed as approximation so the eigenpairs (λ,u) of (2.5). (2.11) is called the Galerkin method determined by the subspace S_h for the approximation of the eigenvalues and eigenvectors of (2.5). We will also sometimes refer to problem (2.11) as the Galerkin approximation of the problem (2.5). (2.11) has a sequence of eigenvalues

$$0 < \lambda_{h,1} \le \lambda_{h,2} \le \cdots \le \lambda_{h,N}, \ N = \dim S_h,$$

and corresponding eigenvectors

$$u_{h,1}, u_{h,2}, \ldots, u_{h,N}$$

which can be chosen to satisfy

$$(2.12)\qquad B_0(u_{h,i},u_{h,j}) = \lambda_{h,i}D(u_{h,i},u_{h,j}) = \delta_{i,j}, \ i,j = 1,\ldots,N.$$

Minimum and minimum-maximum principles analogous to (2.8) and (2.9) hold for the problem (2.11); they are obtained by replacing H by S_h and letting $k = 1,\ldots,N$. We will refer to them by (2.8*) and (2.9*), respectively. Using (2.8) and (2.9) together with (2.8*) and (2.9*) we see immediately that

$$(2.13)\qquad \lambda_k \le \lambda_{h,k}, \ k = 1,2,\ldots,N = \dim S_h.$$

For every $\lambda_{h,k}$ we let

$$M(\lambda_{h,k}) = \{u : u \text{ is an eigenvector of (2.11) corresponding to } \lambda_{h,k}\}. \tag{2.10*}$$

Because of (2.1)-(2.4), 0 is an eigenvalue of neither (2.5) nor (2.11). It will be convenient, however, to introduce the notation $\lambda_0 = \lambda_{h,0} = 0$.

In what will follow we shall assume that the family $\{S_h\}$ satisfies the approximability assumption

$$\inf_{\chi \in S_h} \|u-\chi\|_{B_0} \longrightarrow 0 \quad \text{as} \quad h \longrightarrow 0, \quad \text{for each} \quad u \in H. \tag{2.14}$$

It follows from the variational principles (2.8), (2.9), (2.8*), and (2.9*), and assumption (2.14) that $\lambda_{h,k} \longrightarrow \lambda_k$ as $h \longrightarrow 0$ for each k.

Our analysis employs two functions $\Phi(\lambda)$ and $\Phi_h(\lambda)$ of the non-negative real variable λ which are associated with the eigenvalue of (2.5) and (2.11), respectively. We define

$$\Phi(\lambda) = \inf_{j=1,2,\ldots} \left|1 - \frac{\lambda}{\lambda_j}\right| \tag{2.15}$$

and

$$\Phi_h(\lambda) = \min_{j=1,\ldots,N} \left|1 - \frac{\lambda}{\lambda_{h,j}}\right|. \tag{2.16}$$

It is immediate that the functions are non-negative and continuous in λ and that

$$\Phi(\lambda) = 0 \text{ if and only if } \lambda = \lambda_j \text{ for some } j$$

and

$$\Phi_h(\lambda) = 0 \text{ if and only if } \lambda = \lambda_{h,j} \text{ for some } j.$$

In the following lemmas we give characterization of Φ and

Φ_h which do not involve the eigenvalues λ_j and $\lambda_{h,j}$, respectively. For $0 \le \lambda < \infty$ and $u,v \in H$ define

(2.17) $$B(\lambda,u,v) = B_0(u,v) - \lambda D(u,v).$$

We now have

Lemma 2.1. For all $0 \le \lambda < \infty$,

(2.18) $$\Phi(\lambda) = \inf_{\substack{u\in H\\ \|u\|_{B_0}=1}} \sup_{\substack{v\in H\\ \|u\|_{B_0}=1}} |B(\lambda,u,v)|.$$

Suppose λ_k has multiplicity q, i.e., suppose

$$\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \dots = \lambda_{k+(q-1)} < \lambda_{k+q}.$$

Then, for

(2.19) $$\bar{\lambda}_k = \left(\frac{\lambda_{k-1}^{-1}+\lambda_k^{-1}}{2}\right)^{-1} \le \lambda \le \bar{\lambda}_{k+q} = \left(\frac{\lambda_k^{-1}+\lambda_{k+q}^{-1}}{2}\right)^{-1},$$

we have

(2.20) $$\Phi(\lambda) = |1 - \frac{\lambda}{\lambda_k}|$$

and

(2.21) $$\begin{aligned}\Phi(\lambda) &= \sup_{\substack{v\in H\\ \|v\|_{B_0}=1}} |B(\lambda,u,v)| \\ &= |B(\lambda,u,u)|, \ \forall\, u \in M(\lambda_k) \text{ with } \|u\|_{B_0} = 1.\end{aligned}$$

Proof. For $u,v \in H$ write

$$u = \sum_{j=1}^{\infty} a_j u_j, \quad v = \sum_{j=1}^{\infty} b_j v_j.$$

Then

(2.22) $$B(\lambda,u,v) = \sum_{j=1}^{\infty} a_j b_j (1 - \frac{\lambda}{\lambda_j}).$$

Thus

$$(2.23)\qquad \sup_{\substack{v \in H \\ \|v\|_{B_0}=1}} |B(\lambda,u,v)| = \left[\sum_{j=1}^{\infty} a_j^2 \left(1 - \frac{\lambda}{\lambda_j}\right)^2\right]^{1/2},$$

from which we get

$$\inf_{\substack{u \in H \\ \|u\|_{B_0}=1}} \sup_{\substack{v \in H \\ \|v\|_{B_0}=1}} |B(\lambda,u,v)| = \inf_{j=1,2,\ldots} \left|1 - \frac{\lambda}{\lambda_j}\right| = \Phi(\lambda).$$

This is (2.18). (2.20) follows from the definition of $\Phi(\lambda)$ and an examination of the graphs of $\left|1 - \frac{\lambda}{\lambda_j}\right|$ for $j = 1,2,\ldots$. (2.21) follows from (2.20), (2.22), and (2.23).

In a similar way we have

<u>Lemma 2.2</u>. With H replaced by S_h, λ_k by $\lambda_{h,k}$, and u_h by $u_{h,k}$, Lemma 2.1 holds for $\Phi_h(\lambda)$. (Relationships analogous to those of Lemma 1 will be indicated by an asterisk.)

If λ_k is an eigenvalue of multiplicity g, then $\lambda_{h,k},\ldots,\lambda_{h,k+q-1}$ could be multiple or simple. The graphs of $\Phi(\lambda)$ and $\Phi_h(\lambda)$ is given in Figure 2.1

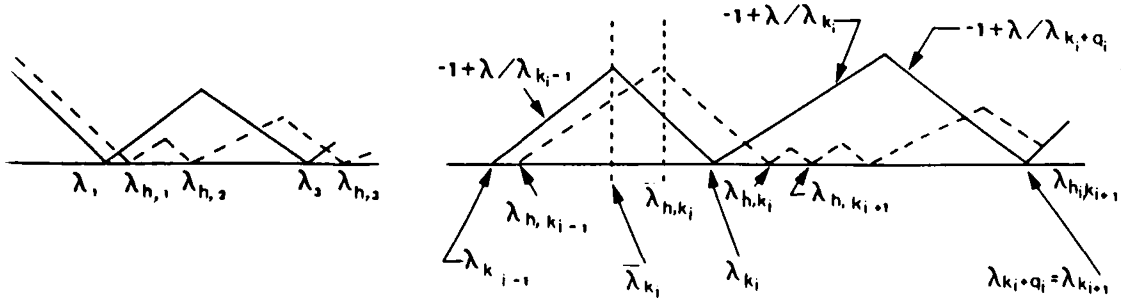

Figure 2.1. The graphs of $\Phi(\lambda)$ and $\Phi_h(\lambda)$.

We end the section with a lemma that expresses a fundamental property of eigenvalue and eigenvector approximation.

<u>Lemma 2.3</u>. Suppose (λ, u) is an eigenpair of (2.5) with $\|u\|_D = 1$, suppose w is any vector in H with $\|w\|_D = 1$, and let $\tilde{\lambda} = B_0(w,w)$. Then

$$\tilde{\lambda} - \lambda = \|w-u\|_{B_0}^2 - \lambda \|w-u\|_D^2. \tag{2.24}$$

(Note that we have assumed u and w are normalized with respect to $\|\cdot\|_D$ here, whereas in (2.6) and (2.12) we assumed u_i and $u_{h,i}$ are normalized with respect to $\|\cdot\|_{B_0}$.)

<u>Proof</u>. By an easy calculation,

$$\begin{aligned} \|w-u\|_{B_0}^2 - \lambda \|w-u\|_D^2 = \|w\|_{B_0}^2 - 2B_0(w,u) + \|u\|_{B_0}^2 \\ - \lambda \|w\|_D^2 + 2\lambda D(w,u) - \lambda \|u\|_D^2. \end{aligned}$$

Then, since

$$\begin{aligned} \|w\|_D = \|u\|_D &= 1, \\ \|w\|_{B_0}^2 &= \tilde{\lambda}, \\ \|u\|_{B_0}^2 &= \lambda, \end{aligned}$$

and

$$B_0(w,u) = \lambda D(w,u),$$

we get the desired result.

3. The main result.

For $i = 1,2,\ldots$ suppose λ_{k_i} is an eigenvalue of (2.5) of multiplicity q_i, i.e., suppose

$$\lambda_{k_i-1} < \lambda_{k_i} = \lambda_{k_i+1} = \cdots = \lambda_{k_i+q_i-1} < \lambda_{k_i+q_i} = \lambda_{k_{i+1}}.$$

Here $k_i = 1$, k_2 is the lowest index of the 2nd distinct eigenvalue, k_3 is the lowest index of the 3rd distinct eigenvalue, etc. Let

$$(3.1) \qquad \varepsilon_{i,j}(h) = \inf_{\substack{u \in M(\lambda_{k_i}) \\ \|u\|_{B_0}=1 \\ B_0(u,u_{h,k_i}) = \cdots = B_0(u,u_{h,k_i+j-2}) = 0}} \ \inf_{\chi \in S_h} \|u-\chi\|_{B_0}, \quad j = 1,\ldots,q_i.$$

The restrictions $B_0(u,u_{h,k_i}) = \cdots = B_0(u,u_{h,k_i+j-2}) = 0$ is considered vacuous if $q_i = 1$. Note that $\varepsilon_{i,1} = \varepsilon_{\lambda_i}$ and $\varepsilon_{i,q_i} = \bar{\varepsilon}_{\lambda_i}$, where ε_{λ_i} and $\bar{\varepsilon}_{\lambda_i}$ are the quantities introduced in Section 1. It is the purpose of this section to estimate the eigenvalue and eigenvector errors for the Galerkin method (2.11) in terms of the approximability quantities $\varepsilon_{i,j}(h)$.

<u>Theorem 3.1</u>. There are constants C and h_0 such that

$$(3.2) \qquad \lambda_{k_i+j-1} - \lambda_{h,k_i+j-1} \le C\varepsilon^2_{i,j}(h), \ \forall\, 0 < h \le h_0, \ j = 1,\ldots,q_i,$$

and

$$(3.3) \qquad \|u_{k_i+j-1} - u_{h,k_i+j-1}\|_{B_0} \le C\varepsilon_{i,j}(h), \ \forall\, 0 < h \le h_0, \ j = 1,\ldots,q_i,$$

$i = 1,2,\ldots$. To be slightly more precise, the eigenvectors $u_1, u_2, \ldots$ of (2.5) can be chosen so that (3.3) holds (as well as (2.6)).

Proof. Overview of the Proof. The complete details of the proof, which proceeds by induction, are given below. Here we provide an overview. In Step A we give the proof for $i = 1$. The proof is very simple in this case and rests entirely on the minimum principle (2.8*) and Lemma 2.3.

The central part of the proof is given in Step B. There we prove the theorem for $i = 2$, proving first the eigenvalue estimate (3.2) and then the eigenvector estimate (3.3). In particular, in Steps B.1 and B.2, estimates (3.2) and (3.3), respectively, are proved for $j = 1$. We further note that the argument used in Step B proves the main inductive step in our proof, yielding the result for $i = \underline{i} + 1$ on the assumption that it is true for $i \le \underline{i}$. To be somewhat more specific, in Step B.1 we prove (3.2) directly for any $i \geq 2$ and in B.2 we prove (3.3) for $i = \underline{i} + 1$ under the assumption that $\|u_{h,j} - u_j\|_{B_0} \longrightarrow 0$ as $h \longrightarrow 0$ for $i \le \underline{i}$ (cf. 3.36).

Details of the Proof. Throughout the proof we use the fact that $\varepsilon_{i,j}(h)$ can also be expressed as

$$\varepsilon_{i,j}(h) = \inf_{\substack{u \in M(\lambda_{k_i}) \\ \|u\|_{B_0} = 1 \\ B_0(u,u_{h,k_i}) = \ldots = B_0(u,u_{h,k_i+j-2}) = 0}} \quad \inf_{\substack{\chi \in S_h^{B_0} \\ B_0(\chi,u_{h,k_i}) = \ldots = B_0(\chi,u_{h,k_i+j-2}) = 0}} \|u - \chi\|_{B_0} . \tag{3.1'}$$

Step A. Here we prove the theorem for $i = 1$.

Step A.1. Suppose λ_{k_1} $(k_1 = 1)$ is an eigenvalue of (2.5) with multiplicity q_1, i.e., suppose $\lambda_1 = \lambda_2 = \ldots = \lambda_{q_1} < \lambda_{q_1+1}$.

In this step we estimate $\lambda_{h,1} - \lambda_1$, the error between λ_1 and the approximate eigenvalue among $\lambda_{h,1},\ldots,\lambda_{h,q_1}$ that is closest to λ_1, i.e., we prove (3.2) for $i = j = 1$. Note that

$$\varepsilon_{1,1}(h) = \inf_{\substack{u \in M(\lambda_1) \\ \|u\|_{B_0}=1}} \inf_{\chi \in S_h} \|u-\chi\|_{B_0}$$

is the error in the approximation by elements of S_h of the most easily approximated eigenvector associated with λ_1.

From the definitions of $\varepsilon_{1,1}(h)$ we see that there is a $\bar{u}_h \in M(\lambda_1)$ with $\|\bar{u}_h\|_{B_0} = 1$ and an $s_h \in S_h$ such that

(3.4) $$\|\bar{u}_h - s_h\|_{B_0} = \varepsilon_{1,1}(h).$$

Let

$$\bar{u} = \frac{\bar{u}_h}{\sqrt{D(\bar{u}_h,\bar{u}_h)}}, \quad \tilde{s}_h = \frac{s_h}{\sqrt{D(s_h,s_h)}} .$$

By the minimum principle (2.8*) we have

(3.5) $$\lambda_{h,1} - \lambda_1 \le B_0(\tilde{s}_h,\tilde{s}_h) - \lambda_1 .$$

Now apply Lemma 2.3 with $(\lambda,u) = (\lambda_1,\bar{u})$, $w = \tilde{s}_h$, and $\sigma = B_0(\tilde{s}_h,\tilde{s}_h)$. This yields

(3.6) $$\begin{aligned} B_0(\tilde{s}_h,\tilde{s}_h) - \lambda_1 &= \|\tilde{s}_h - \bar{u}\|^2_{B_0} - \lambda_1\|\tilde{s}_h - \bar{u}\|^2_D \\ &\le \|\tilde{s}_h - \bar{u}\|^2_{B_0} \le C\|s_h - \bar{u}_h\|^2_{B_0} . \end{aligned}$$

(3.4) - (3.6) yield the desired result.

<u>Step A.2</u>. In this step we prove (3.3) for $i = j = 1$. Let $u_1, u_2, \ldots$ be eigenvectors of (2.5) satisfying (2.6). Write

(3.7) $$u_{h,1} = \sum_{j=1}^{\infty} a_j^{(1)} u_j .$$

From (3.6) and (3.7) we have

$$\left(1-\frac{\lambda_1}{\lambda_{q_1+1}}\right)\sum_{j=q_1+1}^{\infty}\left(a_j^{(1)}\right)^2 \le \left|\sum_{j=q_1+1}^{\infty}\left(a_j^{(1)}\right)^2(1-\lambda_1/\lambda_j)\right|$$

$$= \left|\sum_{j=1}^{\infty}\left(a_j^{(1)}\right)^2(1-\lambda_1/\lambda_j)\right|$$

$$= |B(\lambda_1,u_{h,1},u_{h,1})|$$

$$= |B(\lambda_{h,1},u_{h,1},u_{h,1})+(\lambda_1-\lambda_{h,1})D(u_{h,1},u_{h,1})|$$

$$= (\lambda_{h,1}-\lambda_1)\lambda_{h,1}^{-1}$$

$$\le C\varepsilon_{1,1}^2(h).$$

Hence

(3.9) $$\|u_{h,1} - \sum_{j=1}^{q_1} a_j^{(1)} u_j\|_{B_0} = \left[\sum_{j=q_1+1}^{\infty}\left(a_j^{(1)}\right)^2\right]^{1/2}$$
$$\le C(1-\lambda_1/\lambda_{q_1+1})^{-1/2}\varepsilon_{1,1}(h).$$

Redefining u_1 to be $\dfrac{\sum_{j=1}^{q_1} a_j^{(1)} u_j}{\|\sum_{j=1}^{q_1} a_j^{(1)} u_j\|_{B_0}}$, we easily see that

$\|u_1\|_{B_0} = 1$, so that (2.6) still holds, and

(3.10) $$\|u_{h,1} - u_1\|_{B_0} \le C\varepsilon_{1,1}(h),$$

as desired. Note that u_1 may depend on h.

Step A.3. Suppose $q_1 \ge 2$. From (3.1′) we see that

$$(3.11) \qquad \varepsilon_{1,2}(h) = \inf_{\substack{u \in M(\lambda_1) \\ \|u\|_{B_0}=1 \\ B_0(u,u_{h,1})=0}} \ \inf_{\substack{\chi \in S_h \\ B_0(\chi,u_{h,1})=0}} \|u-\chi\|_{B_0}.$$

Choose $\bar{u}_h \in M(\lambda_1)$ with $\|\bar{u}_h\|_{B_0} = 1$, $B_0(\bar{u}_h,u_{h,1}) = 0$ and $s_h \in S_h$ with $B_0(s_h,u_{h,1}) = 0$ so that

$$(3.12) \qquad \|\bar{u}_h-s_h\|_{B_0} = \varepsilon_{1,2}(h),$$

and let

$$\bar{\bar{u}}_h = \frac{\bar{u}_h}{\sqrt{D(\bar{u}_h,\bar{u}_h)}}, \quad \tilde{s}_h = \frac{s_h}{\sqrt{D(s_h,s_h)}}.$$

Since $B_0(s_h,u_{h,1}) = 0$, from the minimum principle (2.8*), Lemma 2.3, and (3.12), we have

$$(3.13) \qquad \lambda_{h,2} - \lambda_2 \le \|\tilde{s}_h - \bar{\bar{u}}_h\|^2_{B_0} \le C\varepsilon^2_{1,2}(h).$$

This is (3.2) for $i = 1$ and $j = 2$.

Step A.4. In Step A.2 we redefined u_1. Now redefine $u_2,\dots,u_{q_1}$ so that $u_1,\dots,u_{q_1}$ are B_0-orthogonal. Write

$$u_{h,2} = \sum_{j=1}^{\infty} a_j^{(2)} u_j.$$

Now, proceeding as in Step A.2 and using (3.13), we have

$$\begin{aligned}(1 - \lambda_2/\lambda_{q_1+1}) \sum_{j=q_1+1}^{\infty} \left[a_j^{(2)}\right]^2 &\le \left|\sum_{j=1}^{\infty} \left[a_j^{(2)}\right]^2 (1 - \lambda_2/\lambda_j)\right| \\ &= |B(\lambda_2,u_{h,2},u_{h,2})| \\ &= (\lambda_{h,2} - \lambda_2)\lambda_{h,2}^{-1} \\ &\le C\varepsilon^2_{1,2}(h).\end{aligned}$$

Thus

$$(3.14) \qquad \|u_{h,2} - \sum_{j=1}^{q_1} a_j^{(2)} u_j\|_{B_0} \le C\varepsilon_{1,2}(h).$$

But by (3.10),

$$\begin{aligned} a_1^{(2)} &= B_0(u_{h,2},u_1) \\ &= B_0(u_{h,2},u_1-u_{h,1}) \\ &\le \|u_{h,2}\|_{B_0}\|u_1-u_{h,1}\|_{B_0} \qquad (3.15) \\ &\le C\varepsilon_{1,1}(h) \\ &\le C\varepsilon_{1,2}(h). \end{aligned}$$

Combining (3.14) and (3.15) we get

$$\|u_{h,2} - \sum_{j=2}^{q_1} a_j^{(2)} u_j\|_{B_0} \le \|u_{h,2} - \sum_{j=1}^{q_1} a_j^{(2)} u_j\|_{B_0} + \|a_1^{(2)} u_1\|_{B_0}$$

$$\le C\varepsilon_{1,2}(h).$$

Redefining u_2 to be $\dfrac{\sum_{j=2}^{q_1} a_j^{(2)} u_j}{\|\sum_{j=2}^{q_1} a_j^{(2)} u_j\|_{B_0}}$, we see that $\|u_2\|_{B_0} = 1$ and $B_0(u_1,u_2) = 0$, so that (2.6) holds and

$$(3.16) \qquad \|u_{h,2} - u_2\|_{B_0} \le C\varepsilon_{1,2}(h),$$

which is (3.3) for $i = 1$, $j = 2$.

Step A.5. Continuing in the above manner we obtain the proof of (3.2) and (3.3) for $i = 1$ and $j = 1,\ldots,q_1$.

Step B. Here we prove Theorem 3.1. for $i = 2$.

Step B.1. Suppose λ_{k_2} $(k_2 = q_1+1)$ is an eigenvalue of (2.5) of multiplicity q_2. In this step we estimate $\lambda_{h,k_2} - \lambda_{k_2}$, the error between λ_{k_2} and the approximate eigenvalue among $\lambda_{h,k_2}, \ldots, \lambda_{h,k_2+q_2-1}$ that is closest to λ_{k_2}. Note that

$$(3.17) \qquad \varepsilon_{2,1}(h) = \inf_{\substack{u \in M(\lambda_{k_2}) \\ \|u\|_{B_0}=1}} \ \inf_{\chi \in S_h} \|u-\chi\|_{B_0}.$$

Write $\lambda_{k_2-1} = \kappa\lambda_{k_2}$, $\lambda_{h,k_2-1} = \kappa_h\lambda_{h,k_2}$, and $\lambda_{h,k_2} = \varphi_h\lambda_{k_2}$. Then $0 < \kappa < 1$ and $\varphi_h \geq 1$. Let $\psi_h = \dfrac{2\varphi_h}{1+\kappa^{-1}}$. From (2.13) and the definitions of $\bar{\lambda}_k$ and $\bar{\lambda}_{h,k}$ in Lemmas 2.1 and 2.2, respectively, we see that

$$(3.18) \qquad \bar{\lambda}_{k_2} \leq \bar{\lambda}_{h,k_2}.$$

A simple calculation shows that

$$\bar{\lambda}_{k_2} = \frac{2}{1+\kappa^{-1}}\lambda_{k_2}$$

and

$$\bar{\lambda}_{h,k_2} = \frac{2\varphi_h}{1+\kappa_h^{-1}}\lambda_{k_2} = \psi_h\lambda_{k_2}.$$

Hence (3.18) shows that

$$\frac{2}{1+\kappa^{-1}} \leq \psi_h.$$

Since $\lambda_{h,j} \longrightarrow \lambda_j$ as $h \longrightarrow 0$ (see Section 2), we see that $\kappa_h \longrightarrow \kappa$, $\varphi_h \longrightarrow 1$, and $\psi_h \longrightarrow \dfrac{2}{1+\kappa^{-1}}$ as $h \longrightarrow 0$. Thus, noting that $\dfrac{2}{1+\kappa^{-1}} < 1$, we see that we can choose h_0 such that $0 < h < h_0$ implies that

$$(3.19) \qquad \varepsilon \le \frac{2}{1+\varepsilon^{-1}} \le \kappa_h \le \frac{\dfrac{2}{1+\varepsilon^{-1}}+1}{2} = \frac{1+3\varepsilon}{2+2\varepsilon} < 1.$$

From (2.19), (2.20), (2.19*), and (2.20*) (see also Figure 2.1) we get

$$0 \le \Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2}) = \left[1 - \frac{\bar\lambda_{h,k_2}}{\varphi_h \lambda_{k_2}}\right] - \left[1 - \frac{\bar\lambda_{h,k_2}}{\lambda_{k_2}}\right]$$

$$= \varphi_h(1 - \varphi_h^{-1}),$$

and hence, using (3.19), we get

$$\lambda_{h,k_2} - \lambda_{k_2} = \lambda_{k_2}(\varphi_h - 1)$$

$$(3.20) \qquad = \frac{\lambda_{k_2}\left[\Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2})\right]\kappa_h^{-1}}{1 - \left[\Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2})\right]\kappa_h^{-1}}$$

$$\le \frac{\lambda_{k_2}\left[\Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2})\right]\varepsilon^{-1}}{1 - \left[\Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2})\right]\varepsilon^{-1}}$$

$$\le 2\lambda_k\left[\Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2})\right]\varepsilon^{-1},$$

provided that

$$(3.21) \qquad \left[\Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2})\right]\varepsilon^{-1} \le 1/2.$$

We will now show that

$$(3.22) \qquad \Phi_h(\bar\lambda_{h,k_2}) - \Phi(\bar\lambda_{h,k_2}) \le C\varepsilon^2_{2,1}(h), \quad \text{for } h \le h_0,$$

where C depends only on λ_1, λ_{k_2-1}, and λ_{k_2}, and, in particular,

is independent of h, and h_0 depends only on λ_1, λ_{k_2-1}, λ_{k_2}, and the approximability of the eigenvectors in $M(\lambda_{k_2})$ by S_h. As in Step A.1, choose $\bar{u}_h \in M(\lambda_{k_2})$ with $\|\bar{u}_h\|_{B_0} = 1$ and $s_h \in S_h$ such that

$$\|\bar{u}_h - s_h\|_{B_0} = \varepsilon_{2,1}(h). \tag{3.23}$$

We see that s_h is the B_0-orthogonal projection of $\bar{u}_h$ onto S_h, i.e., that

$$B_0(\bar{u}_h - s_h, v) = 0, \ \forall\ v \in S_h. \tag{3.24}$$

Let $w_h = \bar{u}_h - s_h$. Then

$$\|w_h\|_{B_0} = \varepsilon_{2,1}(h) \tag{3.25}$$

and

$$\|s_h\|^2_{B_0} \le \|s_h\|^2_{B_0} + \|w_h\|^2_{B_0} = \|\bar{u}_h\|^2_{B_0} = 1. \tag{3.26}$$

Next we write

$$s_h = c_h\bar{u}_h + e_h,$$

where

$$B_0(\bar{u}_h, e_h) = 0,$$

i.e., we let $c_h\bar{u}_h$ be the B_0-orthogonal projection of s_h onto span $(\bar{u}_h)$. Let $r_h = (1-c_h)\bar{u}_h$. Then

$$w_h = r_h - e_h, \ B(e_h, r_h) = 0,$$

which implies

$$\|e_h\|_{B_0} \le \|w_h\|_{B_0}. \tag{3.27}$$

Furthermore, using (3.24), (3.25), and (3.26) we have

$$c_h = B_0(s_h, \bar{u}_h) \le \|s_h\|_{B_0} \|\bar{u}_h\|_{B_0} \le 1$$

and

$$\begin{aligned} c_h = B_0(s_h, \bar{u}_h) &= B_0(\bar{u}_h - w_h, \bar{u}_h) \\ &= \|\bar{u}_h\|_{B_0}^2 - B_0(w_h, \bar{u}_h) \\ &\ge 1 - \|w_h\|_{B_0} \\ &= 1 - \varepsilon_{2,1}(h) \\ &> 0, \quad \text{for } h < h_0, \end{aligned}$$

with h_0 sufficiently small. Thus we can assume

$$0 < c_h \le 1.$$

Also,

$$\|r_h\|_{B_0} = \|w_h\|_{B_0}^2. \tag{3.28}$$

To see this we refer to Figure 3.1 and note that

$$\|r_h\|_{B_0} = \|w_h\|_{B_0} \cos \alpha$$

and

$$\|w_h\|_{B_0} = \|\bar{u}_h\|_{B_0} \sin \beta = \cos \alpha .$$

From (3.28) an the definition of r_h we have

$$c_h = 1 - \|w_h\|_{B_0}^2 . \tag{3.29}$$

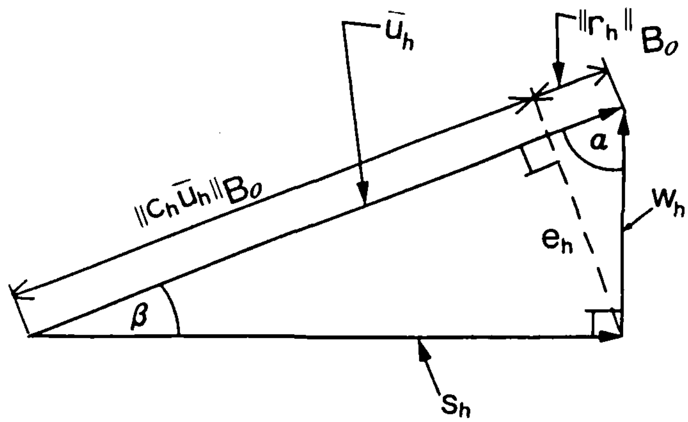

Figure 3.1. Configuration of r_h, w_h, $\bar{u}_h$, α, and β.

Assume now that $u_k = \bar{u}_h$ (redefining u_k, if necessary). Write

$$e_h = \sum_{j=1}^{\nu} q_j u_j, \quad v = \sum_{j=1}^{\nu} b_j u_j.$$

Then

$$\begin{aligned}
B(\bar{\lambda}_{h,k_2}, s_h, v) &= c_h B_0(\bar{u}_h, v) + B_0(e_h, v) \\
&\quad - \bar{\lambda}_{h,k_2} c_h D(\bar{u}_h, v) - \bar{\lambda}_{h,k_2} D(e_h, v) \\
&= c_h b_{k_2} + \sum_{\substack{j=1 \\ \neq k_2}}^{\nu} q_j b_j - \bar{\lambda}_{h,k_2} \lambda_{k_2}^{-1} c_h b_{k_2} \\
&\quad - \bar{\lambda}_{h,k_2} \sum_{\substack{j=1 \\ \neq k_2}}^{\nu} q_j b_j \lambda_j^{-1} \\
&= c_h b_{k_2}(1 - \bar{\lambda}_{h,k_2} \lambda_{k_2}^{-1}) + \sum_{\substack{j=1 \\ \neq k_2}}^{\nu} q_j b_j (1 - \bar{\lambda}_{h,k_2} \lambda_j^{-1}),
\end{aligned}$$

and hence

$$\sup_{\substack{v \in S_h \\ \|v\|_{B_0}=1}} |B(\bar{\lambda}_{h,k_2}, s_h, v)| \le \sup_{\substack{v \in H \\ \|v\|_{B_0}=1}} |B(\bar{\lambda}_{h,k}, s_h, v)|$$

$$(3.30) \qquad = \Big[c_h^2\Big(1 - \bar{\lambda}_{h,k_2}\lambda_{k_2}^{-1}\Big)^2 + \sum_{\substack{j=1 \\ \ne k_2}}^{\infty} q_j^2\Big(1 - \bar{\lambda}_{h,k_2}\lambda_j^{-1}\Big)^2\Big]^{1/2}.$$

Combining (3.27), (3.29), and (3.30), we get

$$\sup_{\substack{v \in S_h \\ \|v\|_{B_0}=1}} |B(\bar{\lambda}_{h,k_2}, s_h, v)| \le \Big[(1 - \|w_h\|_{B_0}^2)^2(1 - \bar{\lambda}_{h,k_2}\lambda_{k_2}^{-1})^2$$

$$+ \sup_{\substack{j=1,2,\ldots \\ \ne k_2}} (1 - \bar{\lambda}_{h,k_2}\lambda_k^{-1})^2 \sum_{j=1}^{\infty} q_j^2\Big]^{1/2}$$

$$\le \Big[(1 - \|w_h\|_{B_0}^2)^2(1 - \bar{\lambda}_{h,k_2}\lambda_{k_2}^{-1})^2$$

$$+ \sup_{\substack{j=1,2,\ldots \\ \ne k_2}} (1 - \bar{\lambda}_{h,k_2}\lambda_{k_2}^{-1})^2\|e_h\|_{B_0}^2\Big]^{1/2}$$

$$= \Big[(1 - \|w_h\|_{B_0}^2)^2\phi^2(\bar{\lambda}_{h,k_2})$$

$$+ \sup_{\substack{j=1,2,\ldots \\ \ne k_2}} (1 - \bar{\lambda}_{h,k_2}\lambda_{k_2}^{-1})^2\|w_h\|_{B_0}^2\Big]^{1/2}$$

$$= Q.$$

Thus, using (3.25) and (3.26), we have

$$\Phi_h(\bar{\lambda}_{h,k_2}) = \frac{Q}{\|s_h\|_{B_0}}$$

$$\leq \Big[(1 - \|w_h\|^2_{B_0})\Phi^2(\bar{\lambda}_{h,k_2})$$

$$+ \sup_{\substack{j=1,2,\ldots \\ \neq k_2}} (1 - \bar{\lambda}_{h,k_2}\lambda_k^{-1})^2 \|w_h\|^2_{B_0}(1 - \|w_h\|^2_{B_0})^{-1}\Big]^{1/2}$$

$$= [\Phi^2(\bar{\lambda}_{h,k_2}) + \varepsilon^2_{2,1}(h)]^{1/2},$$

and hence

$$\Phi_h(\bar{\lambda}_{h,k_2}) - \Phi(\bar{\lambda}_{h,k_2}) \leq \frac{R\varepsilon^2_{2,1}(h)}{2\Phi(\bar{\lambda}_{h,k_2})}, \tag{3.31}$$

where

$$R = \sup_{\substack{j=1,2,\ldots \\ \neq k_2}} (1 - \bar{\lambda}_{h,k_2}\lambda_{k_2}^{-1})(1 - \varepsilon^2_{2,1}(h))^{-1} - \Phi^2(\bar{\lambda}_{h,k_2}).$$

One can easily show that $\frac{R}{2\Phi(\bar{\lambda}_{h,k_2})}$ is bounded independent of h; in fact, using (2.20) and (3.19), one can show that

$$\frac{R}{2\Phi(\bar{\lambda}_{h,k_2})} \leq \frac{2\lambda_{k_2}}{2\lambda_1(1-\varepsilon)[1-\varepsilon^2_{2,1}(h)]} \leq \frac{\lambda_{k_2}}{\lambda_1(1-\varepsilon)}, \tag{3.32}$$

provided $\varepsilon^2_{2,1}(h) < 1/2$, i.e., provided $h < h_0$ for h_0 sufficiently small. Hence, combining (3.31) and (3.32) we get

$$\Phi_h(\bar{\lambda}_{h,k_2}) - \Phi(\bar{\lambda}_{h,k_2}) \leq \frac{\lambda_{k_2}}{\lambda_1(1-\varepsilon)}\varepsilon^2_{2,1}(h), \tag{3.33}$$

which is (3.22). Combining (3.20), (3.21), and (3.33) we have

$$\lambda_{h,k_2} - \lambda_{k_2} \leq C\varepsilon^2_{2,1}(h), \quad \text{for } h \leq h_0, \tag{3.34}$$

where

(3.35) $$C = \frac{2\lambda_{k_2}^2}{\lambda_1 \epsilon (1-\epsilon)} .$$

This is (3.2) for $i = 2$, $j = 1$.

<u>Comment on Inequality (3.34)</u>. C in (3.34) clearly depends on λ_1, λ_{k_2-1}, and λ_{k_2}, but is independent of h. Note that if we were considering a family of problems depending on a parameter τ, we could bound $C = C(\tau)$ above, independent of τ, provided $\lambda_{k_2} = \lambda_{k_2}(\tau)$ was bounded above, $\lambda_1 = \lambda_1(\tau)$ was bounded away from 0 and $\epsilon = \epsilon(\tau)$ was bounded away from 0 and 1. It follows from (3.31) and (3.33), that (3.34) is valid for $h \le h_0$, where h_0 depends only on λ_1, λ_{k_2-1}, λ_{k_2}, and the approximability of the eigenvectors u_j, $j = k_2,\ldots,k_2+q_2-1$, by S_h. For a family of problems, $h_0(\tau)$ could be bounded away from 0 if $\lambda_1(\tau)$ was bounded away from 0, $\lambda_{k_2}(\tau)$ was bounded above, $\epsilon(\tau)$ was bounded away from 0 and 1, and the eigenvectors $u_j = u_j(\tau)$, $j = k_2,\ldots,k_2+q_2-1$, could be approximated by S_h, uniformly in τ.

<u>Step B.2</u>. Suppose, as in Step B.1, that λ_{k_2} has multiplicity q_2. We have shown in Step A.5 that we can choose the eigenvectors $u_1, u_2, \ldots$ of (2.5) so that (2.6) holds and so that

(3.36) $$\|u_{h,j} - u_j\|_{B_0} \le C\epsilon_{1,j}(h), \quad j = 1,\ldots,q_1 = k_2-1.$$

Write

(3.37) $$u_{h,k_2} = \sum_{j=1}^{\infty} a_j^{(k_2)} u_j .$$

From (3.37) we have

$$\left|\sum_{j=1}^{\infty}\left[a^{(k_2)}\right]^2(1-\lambda_{k_2}/\lambda_j)\right| = |B(\lambda_{k_2},u_{h,k_2},u_{h,k_2})|$$

$$= |B(\lambda_{h,k_2},u_{h,k_2},u_{h,k_2})$$

$$+ (\lambda_{k_2}-\lambda_{h,k_2})D(u_{h,k_2},u_{h,k_2})|$$

$$= (\lambda_{h,k_2}-\lambda_{k_2})\lambda_{h,k_2}^{-1},$$

which, together with (3.34), yields

$$(3.38)\quad \left|\sum_{j=1}^{k_2-1}\left[a^{(k_2)}\right]^2(1-\lambda_{k_2}/\lambda_j)\right| + \sum_{j=k_2+q_2}^{\infty}\left[a^{(k_2)}\right]^2(1-\lambda_{k_2}/\lambda_j)|$$

$$\le C\varepsilon_{2,1}^2(h).$$

Note that the first term inside the absolute value is negative and the second is positive. In addition

$$C_1 \le |1-\lambda_{k_2}/\lambda_j| \le C_2,\ \forall_j = k_2,\ k_2+1,\ \ldots,\ k_2+q_2-1,$$

with C_1, C_2 positive numbers. Hence from (3.38) we obtain

$$(3.39)\quad \sum_{j=1}^{k_2-1}\left[a^{(k_2)}\right]^2 \le D_1\varepsilon_{2,1}^2(h) + D_2\sum_{j=k_2+q_2}^{\infty}\left[a^{(k_2)}\right]^2$$

and

$$(3.40)\quad \sum_{j=k_2+q_2}^{\infty}\left[a^{(k_2)}\right]^2 \le D_3\varepsilon_{2,1}^2(h) + D_3\sum_{j=1}^{k_2-1}\left[a^{(k_2)}\right]^2.$$

Write

$$(3.41)\quad u_{h,i} - u_i = \sum_{j=1}^{\infty} b_{i,j}u_j,\ i = 1,\ldots,k_2-1 = q_1.$$

Then, by (3.36),

$$(3.42)\qquad \sum_{j=1}^{\lambda} b_{i,j}^2 = \|u_{h,i} - u_i\|_{B_0}^2 \le C\varepsilon_{2,i}^2(h),\ i = 1,\ldots,k_2-1.$$

Next we wish to find constants $\alpha_1,\ldots,\alpha_{k_2-1}$ so that

$$(3.43)\qquad B_0\Big(u_i, \sum_{j=1}^{k_2-1} \alpha_j u_{h,j}\Big) = a_i^{(k_2)},\ i = 1,\ldots,k_2-1.$$

Using (3.41), these equations can be written as

$$(3.44)\qquad \begin{aligned} B_0\Big[u_i, \sum_{j=1}^{k_2-1} \Big(\alpha_j u_j + \alpha_j \sum_{\ell=1}^{\lambda} b_{j,\ell} u_\ell\Big)\Big] &= \alpha_i + \sum_{j=1}^{k_2-1} b_{j,i}\alpha_j \\ &= a_i^{(k_2)},\ i = 1,\ldots,k_2-1. \end{aligned}$$

Since (2.14) implies $\varepsilon_{2,1}^2(h)\longrightarrow 0$ as $h\longrightarrow 0$, from (3.42) we see that the $b_{j,i}$ are small for $h \le h_0$, with h_0 sufficiently small, and hence the system (3.44) is uniquely solvable, and, moreover, there is a constant L, depending only on k_2, such that

$$(3.45)\qquad \left[\sum_{j=1}^{k_2-1} \alpha_i^2\right]^{1/2} \le L\left[\sum_{j=1}^{k_2-1} \left(a_i^{(k_2)}\right)^2\right]^{1/2}.$$

Now, from (3.36) we obtain

$$\begin{aligned} |a_j^{(k_2)}| &= |B_0(u_{h,k_2}, u_j)| \\ &= |B_0(u_{h,k_2}, u_j - u_{h,j})| \\ &\le \|u_{h,k_2}\|_{B_0} \|u_j - u_{h,j}\|_{B_0} \\ &= \|u_j - u_{h,j}\|_{B_0} \\ &\le C\varepsilon_{1,j}(h),\ j = 1,\ldots,k_2-1. \end{aligned}$$

Letting

$$\rho_{k_2}^2(h) = \sum_{j=1}^{k_2-1} \varepsilon_{i,j}^2(h), \tag{3.46}$$

we see that

$$\left[\sum_{j=1}^{k_2-1} \left(a_j^{(k_2)}\right)^2\right]^{1/2} \le C\rho_{k_2}(h), \tag{3.47}$$

and thus, from (3.45)

$$\left(\sum_{j=1}^{k_2-1} \alpha_i^2\right)^{1/2} \le LC\rho_{k_2}(h) \le C\rho_{k_2}(h). \tag{3.48}$$

Now let

$$\iota = u_{h,k_2} - \sum_{j=1}^{k_2-1} \alpha_j u_{h,j}. \tag{3.49}$$

Then $\iota \in S_h$. Furthermore, from (3.43) and (3.44) we get

$$B_0(u_i,\iota) = \begin{cases} 0, & i \le k_2 - 1 \\ a_i^{(k_2)} - \displaystyle\sum_{j=1}^{k_2-1} \alpha_j b_{j,i}, & i \ge k_2 \end{cases} \tag{3.50}$$

From (3.48) and (3.49),

$$\begin{aligned} \left|\|\iota\|_{B_0} - 1\right| &= \left|\|\iota\|_{B_0} - \|u_{h,k_2}\|_{B_0}\right| \\ &\le \|\iota - u_{h,k_2}\|_{B_0} \\ &\le \left(\sum_{j=1}^{k_2-1} \alpha_j^2\right)^{1/2} \\ &\le C\rho_{k_2}(h). \end{aligned} \tag{3.51}$$

Using (3.34), (2.21*), (3.50), and (3.51), and the fact that $\rho_{k_2}(h) \longrightarrow 0$ as $h \longrightarrow 0$, we get

$$
\begin{aligned}
C\varepsilon^2_{2,1}(h) &\le \frac{\lambda_{h,k_2} - \lambda_{k_2}}{\lambda_{h,k_2}} \\
&= \Phi_h(\lambda_{k_2}) \\
&\le B\left(\lambda_{k_2}, u_{h,k_2}, \frac{\iota}{\|\iota\|_{B_0}}\right) \\
&= C'\left[\sum_{\ell=k_2+q_2}^{\infty} a_\ell^{(k_2)}\left(a_\ell^{(k_2)} - \sum_{i=1}^{k_2-1} \alpha_i b_{i,\ell}\right)\left(1 - \frac{\lambda_{k_2}}{\lambda_\ell}\right)\right],
\end{aligned}
\tag{3.52}
$$

where $C' > 0$ and is independent of h. Combining (3.42), (3.45), (3.4), and (3.52) we obtain

$$
\begin{aligned}
\sum_{\ell=k_2+q_2}^{\infty}\left(a_\ell^{(k_2)}\right)^2 &\le C\Bigg[\varepsilon^2_{2,1}(h) \\
&\quad + \sum_{\ell=k_2+q_2}^{\infty}|a_\ell^{(k_2)}|\sum_{i=1}^{k_2-1}|\alpha_i||b_{i,\ell}|\Bigg] \\
&\le C\left[\varepsilon^2_{2,1}(h) + \sum_{i=1}^{k_2-1}|\alpha_i|\sum_{\ell=k_2+q_2}^{\infty}|a_\ell^{(k_2)}||b_{i,\ell}|\right] \\
&\le C\Bigg[\varepsilon^2_{2,1}(h) + \sum_{i=1}^{k_2-1}|\alpha_i|\left(\sum_{\ell=k_2+q_2}^{\infty}|a_\ell^{(k_2)}|^2\right)^{1/2} \\
&\qquad \left(\sum_{\ell=k_2+q_2}^{\infty}|b_{i,\ell}|^2\right)^{1/2}\Bigg]
\end{aligned}
\tag{3.53}
$$

$$\le C\left[\varepsilon_{2,1}^{2}(h) + \sum_{i=1}^{k_2-1} |\alpha_i| \left(\sum_{\ell=k_2+q_2}^{\infty} |a_\ell^{(k_2)}|^2\right)^{1/2} \times \max_{i=1,\ldots,k_2-1} \varepsilon_{1,i}(h)\right]$$

$$\le C\left[\varepsilon_{2,1}^{2}(h) + \varepsilon_{1,k_2-1}(h)\sqrt{k_2-1}\left(\sum_{i=1} |\alpha_i|^2\right)^{1/2} \times \left(\sum_{\ell=k_2+q_2}^{\infty} |a_\ell^{(k_2)}|^2\right)^{1/2}\right]$$

$$\le C\left[\varepsilon_{2,1}^{2}(h)+ \varepsilon_{1,k_2-1}(h)\sqrt{k_2-1}\, L\left(\sum_{i=1}^{k_2-1} \left[a_i^{(k_2)}\right]^2\right)^{1/2} \times \left(\sum_{\ell=k_2+q_2}^{\infty} |a_\ell^{(k_2)}|^2\right)^{1/2}\right]$$

$$\le C\left[\varepsilon_{2,1}^{2}(h)+ \varepsilon_{1,k_2-1}(h)\left(\sum_{i=1}^{k_2-1} \left[a_i^{(k_2)}\right]^2\right)^{1/2} \times \left(\sum_{\ell=k_2+q_2}^{\infty} |a_\ell^{(k_2)}|^2\right)^{1/2}\right].$$

(3.53) is a quadratic inequality in $\left[\sum_{\ell=k_2+q_2}^{\infty}\left[a_\ell^{(k_2)}\right]^2\right]^{1/2}$ whose solution yields

(3.54) $$\sum_{\ell=k_2+q_2}^{\infty}\left[a_\ell^{(k_2)}\right]^2 \le C\varepsilon_{1,k_2-1}^{2}(h)\sum_{i=1}^{k_2-1}\left[a_i^{(k_2)}\right]^2 + C\varepsilon_{2,1}^{2}(h).$$

Combining (3.39) and (3.54) we get

$$\sum_{i=1}^{k_2-1}\left[a_i^{(k_2)}\right]^2 \le D_1\varepsilon_{2,1}^2(h) + D_2C\varepsilon_{1,k_2-1}^2(h)\sum_{i=1}^{k_2-1}\left[a_i^{(k_2)}\right]^2 + D_2C\varepsilon_{2,1}^2(h),$$

and thus, since $\varepsilon_{1,k_2-1}(h)$ is small for h small,

$$\sum_{i=1}^{k_2-1}\left[a_i^{(k_2)}\right]^2 \le D_5\varepsilon_{2,1}^2(h). \tag{3.55}$$

Next, combining (3.40) and (3.55), we get

$$\sum_{\ell=k_2+q_2}^{\lambda}\left[a_\ell^{(k_2)}\right]^2 \le D_6\varepsilon_{2,1}^2(h). \tag{3.56}$$

Finally, from (3.37), (3.55), and (3.56), we have

$$\|u_{h,k_2}-\sum_{j=k_2}^{k_2+q_2-1}a_j^{(k_2)}u_j\|_{B_0} = \left[\sum_{j=1}^{k_2-1}\left[a_i^{(k_2)}\right]^2+\sum_{j=k_2+q_2}^{\lambda}\left[a_j^{(k_2)}\right]^2\right]^{1/2}$$

$$\le C\varepsilon_{2,1}(h).$$

Redefining u_{k_2} to be $\dfrac{\sum_{j=k_2}^{k_2+q_2-1} a_j^{(k_2)}u_j}{\|\sum_{j=k_2}^{k_2+q_2-1} a_j^{(k_2)}u_j\|_{B_0}}$, we see that $\|u_{k_2}\|_{B_0} =$ 1, so that (2.6) holds, and

$$\|u_{h,k_2}-u_{k_2}\|_{B_0} \le C\varepsilon_{2,1}(h). \tag{3.57}$$

This is (3.3) for $i = 2$, $j = 1$.

Comment on Estimate (3.57). In the proof of (3.57) we used (3.36), which was proved in Step A. A careful examinations of the proof of (3.57) show that we that we did not use the full strength of (3.36), but only the weaker fact that $\|u_{h,j} - u_j\|_{B_0} \longrightarrow 0$ as $h \longrightarrow 0$ for $j \le k_2 - 1$. (Cf. the Overview of the Proof.)

Step B.3. Suppose $q_2 \ge 2$. In Step B.1 we estimated $\lambda_{h,k_2} - \lambda_{k_2}$. In this step we estimate $\lambda_{h,k_2+1} - \lambda_{k_2+1}$.

We proceed by modifying problems (2.5) and (2.11) by restricting them to the space

$$H^{h,k_2} = \{u \in H : B_0(u, u_{h,k_2}) = 0\}$$

and

$$S_h^{h,k_2} = \{u \in S_h : B_0(u, u_{h,k_2}) = 0\},$$

respectively, i.e., we consider the problems $(2.5^{h,k_2})$ and $(2.11^{h,k_2})$ obtained when H and S_h are replaced by H^{h,k_2} and S_h^{h,k_2} in (2.5) and (2.11), respectively. $(2.11^{h,k_2})$ has the same eigenpairs $(\lambda_{h,j}, u_{h,j})$ as does (2.11) except that the pair $(\lambda_{h,k_2}, u_{h,k_2})$ is eliminated. $(2.5^{h,k_2})$ has eigenpairs $(\lambda_j^{h,k_2}, u_j^{h,k_2})$ which in general depend on h. Nevertheless,

$$\lambda_{k_2+\ell}^{h,k_2} = \lambda_{k_2+\ell}, \quad \ell = 0, \ldots, q_2 - 2, \tag{3.58}$$

i.e., λ_{k_2}, the eigenvalue under consideration, is an eigenvalue of multiplicity $q_2 - 1$ for problem $(2.5^{h,k_2})$. Its eigenspace is

$$M^{h,k_2}(\lambda_{k_2}) = \{u \in M(\lambda_{k_2}) : B_0(u,u_{h,k_2}) = 0\}.$$

We can now apply the argument used in Step B.1 to problems $(2.5^{h,k_2})$ and $(2.11^{h,k_2})$ and we obtain (cf. (3.34))

(3.59) $$\lambda_{h,k_2+1} - \lambda_{k_2+1} \le C\varepsilon^2_{2,2}(h), \quad \text{for} \quad h < h_0.$$

Since u_{h,k_2} depends on h, the problems $(2.5^{h,k_2})$ and $(2.11^{h,k_2})$ depend on h. It follows from the Comment on Inequality (3.34) with $\tau = h$ that we can apply the argument in Step B.1 obtaining C and h_0 that are independent of h. To see this, note that $\lambda^{h,k_2}_{k_2} = \lambda_{k_2}$, by (3.58), $\lambda^{h,k_2}_1 \ge \lambda_1$, by the minimum principle, and $\varepsilon^{h,k_2} = \lambda^{h,k_2}_{k_2-1}/\lambda^{h,k_2}_{k_2} \longrightarrow \lambda_{k_2-1}/\lambda_{k_2} = \varepsilon$ as $h \longrightarrow 0$, by the minimum-maximum principle and the fact that $u_{h,k_2} \longrightarrow u_{k_2}$ (cf. (3.57)), and hence $\lambda^{h,k_2}_{k_2}$ is bounded from above, λ^{h,k_2}_1 is bounded from below, and ε^{h,k_2} is bounded away from 0 and 1. Then observe that the eigenvectors in $M^{h,k_2}(\lambda_{k_2})$ can be approximated by S_h, uniformly in h. Note that in Step B.1 we also used the fact that $\lambda_{h,j} \longrightarrow \lambda_j$ as $h \longrightarrow 0$ for $j = k_2 - 1$ and k_2. It is easy to see that the corresponding fact is true in the present context.

<u>Step B.4</u>. Suppose $q_2 \ge 2$ as in Step B.3. Here we show that u_{k_2+1} can be chosen so that $\|u_{h,k_2+1} - u_{k_2+1}\|_{B_0} \le C\varepsilon_{2,2}(h)$. We know that

$$(3.60) \qquad \|u_{h,j} - u_j\|_{B_0} \le \begin{cases} C\varepsilon_{1,j}(h), & j = 1,\ldots,q_1 \\ C\varepsilon_{2,1}(h), & j = q_1+1 = k_2 \end{cases},$$

(cf. (3.16), (3.14), and (3.57)). Assume that $u_{k_2+1},\ldots,u_{k_2+q_2-1}$ have been redefined so that (2.6) holds. Write

$$(3.61) \qquad u_{h,k_2+1} = \sum_{j=1}^{\ell} a_j^{(k_2+1)} u_j .$$

If we apply the argument used in Step B.2 to u_{h,k_2+1}, i.e., if we let k_2 be replaced by k_2+1 and use (3.59) instead of (3.34), we obtain

$$(3.62) \qquad \|u_{h,k_2+1} - \sum_{j=k_2}^{k_2+q_2-1} a_j^{(k_2)} u_j\|_{B_0} \le C\varepsilon_{2,2}(h).$$

But, by (3.60),

$$\begin{aligned} |a_{k_2}^{(k_2)}| &= |B_0(u_{h,k_2},u_{k_2})| \\ &= |B_0(u_{h,k_2},u_{k_2} - u_{h,k_2})| \\ &\le \|u_{k_2} - u_{h,k_2}\|_{B_0} \\ &\le C\varepsilon_{2,1}(h) \\ &\le C\varepsilon_{2,2}(h) \end{aligned}$$

and hence

$$\|u_{h,k_2+1} - \sum_{j=k_2+1}^{k_2+q_2-1} a_j^{(k_2)} u_j\|_{B_0} \le C\varepsilon_{2,2}(h).$$

Redefining u_{k_2+1} to be $\dfrac{\sum_{j=k_2+1}^{k_2+q_2+1} a_j^{(k_2)} u_j}{\left\| \sum_{j=k_2+1}^{k_2+q_2-1} a_j^{(k_2)} u_j \right\|_{B_0}}$, we see that

$\|u_{k_2+1}\|_{B_0} = 1$, $B_0(u_{k_2+1}, u_j) = 0$, $j = 1,\ldots,k_2$, so that (2.6) holds, and

$$\|u_{h,k_2+1} - u_{k_2+1}\|_{B_0} \le C\varepsilon_{2,2}(h), \tag{3.63}$$

which is (3.3) for $i = j = 2$.

Step B.5. Continuing in this manner we prove (3.2) and (3.3) for $i = 2$ and $j = 1,\ldots,q_2$.

Step C. Repeating the argument in B we get (3.2) and (3.3) for $i = 3,4,\ldots$. This completes the proof.

Remark 3.1. It is possible to use an alternate argument in Step B.1 if we introduce the so-called Riesz formulas for the spectral projections associated with an operator. We suppose the space H and the bilinear forms B_0 and D have been complexified in the usual manner. Let P_{λ_2} and P_{h,λ_2} be the B_0-orthogonal projections of H onto $M(\lambda_{k_2})$ and $\sum_{i=0}^{q_2-1} M(\lambda_{h,k_2+i})$, the direct sum of the eigenspaces $M(\lambda_{h,k_2+i})$, $i = 0,\ldots,q_2-1$, respectively. Introduce next the operators T, $T_h : H \longrightarrow H$ defined by

$$\begin{cases} Tf \in H \\ B_0(Tf, v) = D(f, v), \ \forall\, v \in H \end{cases}$$

and

$$\begin{cases} T_h f \in S_h \\ B_0(T_h f, v) = D(f,v), \ \forall \ v \in S_h \end{cases}.$$

It follows from (2.1), (2.2), and (2.3) that T and T_h are defined and compact on H. Furthermore

$$\|(T-T_h)f\|_{B_0} \le C \inf_{\chi \in S_h} \|Tf - \chi\|_{B_0}. \tag{3.64}$$

It is immediate that (λ, u) is an eigenpair of (2.5) if and only if $(\mu = \lambda^{-1}, u)$ is an eigenpair of T. Likewise (λ_h, u_h) is an eigenpair of (2.11) if and only if $(\mu_h = \lambda_h^{-1}, u_h)$ is an eigenpair of T_h. As a consequence of (2.14), $T_h \longrightarrow T$ in the operator norm associated with $\|\cdot\|_{B_0}$. Let Γ be a circle in the complex plane centered at $\mu_{k_2} = \lambda_{k_2}^{-1}$, enclosing no other eigenvalues of T. Then for h sufficiently small, Γ will contain the eigenvalues $\mu_{h,k_2+i} = \lambda_{h,k_2+i}^{-1}$, $i = 0,\ldots,q_2-1$, of T_h. Also, P_{λ_2} and P_{h,λ_2}, which are referred to as spectral projections associated with T and μ_{λ_2} and T_h and μ_{h,k_2+i}, $i = 0,\ldots,q_2-1$, respectively, can be written as

$$P_{\lambda_2} = \frac{1}{2\pi i} \int_\Gamma (z-T)^{-1} dz \tag{3.65}$$

and

$$P_{h,\lambda_2} = \frac{1}{2\pi i} \int_\Gamma (z-T_h)^{-1} dz. \tag{3.66}$$

These are the Riesz formulas. With these formulas we can derive an eigenvector error estimate which will lead to the eigenvalue estimate (3.34).

Let $u \in M(\lambda_2)$ with $\|u\|_0 = 1$. Then $v_h = P_{h,\lambda_2} u \in \sum_{i=0}^{q_2-1} M(\lambda_{h,k_2+i})$ and from the formulas (3.65) and (3.66) we obtain

$$
\begin{aligned}
\|u - v_h\|_{B_0} &= \|(P_{\lambda_{k_2}} - P_{h,\lambda_{k_2}})u\|_{B_0} \\
&= \left\|\frac{1}{2\pi i}\int_\Gamma (z-T_h)^{-1}(T-T_h)(z-T)^{-1}u\,dz\right\| \\
&= \left\|\frac{1}{2\pi i}\int_\Gamma (z-T_h)^{-1}(T-T_h)\frac{u}{z-\mu}\,dz\right\| \qquad (3.67) \\
&\le \frac{1}{2\pi}\,[2\pi\,\mathrm{rad}(\Gamma)]\sup_{z\in\Gamma}\|(z-T_h)^{-1}\| \times \frac{1}{\mathrm{rad}(\Gamma)}\|(T-T_h)u\|_{B_0} \\
&= (-\mu_{k_2} + \mathrm{rad}(\Gamma) + \mu_{h,k_2+q_2-1})^{-1}\|(T-T_h)u\|_{B_0} \\
&\le C\|(T-T_h)u\|_{B_0}.
\end{aligned}
$$

(3.64) and (3.67) yield

$$
\begin{aligned}
\|u-v_h\|_{B_0} &\le C \inf_{\phi\in S_h}\|Tu-\phi\|_{B_0} \\
&= C \inf_{\phi\in S_h}\|\mu u-\phi\|_{B_0} \qquad (3.68) \\
&\le C \inf_{\phi\in S_h}\|u-\phi\|_{B_0}.
\end{aligned}
$$

This is an eigenvector estimate; it shows that starting from any $u \in M(\lambda_2)$ with $\|u\|_0 = 1$ we can construct a $v_h = v_h(u) \in \sum_{i=1}^{q_2-1} M(\lambda_{h,k_2+i})$ that is close to u. We now use (3.68) to prove (3.34).

By the minimum principle (2.8*) we have

$$(3.69) \qquad \lambda_{h,k_2} - \lambda_{k_2} = \inf_{\substack{v \in S_h \\ \|v\|_D = 1 \\ B_0(v,u_{h,i})=0, \\ i=1,\ldots,k_2-1}} B_0(v,v) - \lambda_{k_2}.$$

Since $v_h = \sum_{i=1}^{q_2-1} \oplus M_{\lambda_{h,k_2+i}}$ we know that $B_0(v_h,u_{h,i}) = 0$, $i = 1,\ldots,k_2-1$. Thus, from (3.69) we find

$$\lambda_{h,k_2} - \lambda_{k_2} \le B_0\left(\frac{v_h}{\|v_h\|_D}, \frac{v_h}{\|v_h\|_D}\right) - \lambda_{k_2}.$$

Combining this with Lemma 2.3 and (3.68) we obtain

$$\lambda_{h,k_2} - \lambda_{k_2} \le \left\|\frac{v_h}{\|v_h\|_D} - \frac{u}{\|u\|_D}\right\|^2_{B_0} - \lambda_{k_2}\left\|\frac{v_h}{\|v_h\|_D} - \frac{u}{\|u\|_D}\right\|_D$$

$$\le C\|v_h - u\|^2_{B_0}$$

$$\le C \inf_{\chi \in S_h} \|u-\chi\|^2_{B_0}.$$

Since this is valid for any $u \in M(\lambda_{k_2})$ with $\|u\|_D = 1$, we see that

$$\lambda_{h,k_2} - \lambda_{k_2} \le C \inf_{\substack{u \in M(\lambda_{k_2}) \\ \|u\|_{B_0}=1}} \inf_{\chi \in S_h} \|u-\chi\|^2_{B_0}$$

$$= C\varepsilon^2_{2,1}(h),$$

which is (3.34). We note that the proof given here rests on equation (2.24) in Lemma 2.3 and employs formulas (3.65) and (3.66) to construct $v_h = v_h(u)$ that is B_0-orthogonal to $u_{h,i}$, $i = 0,\ldots,k_2-1$, and satisfies

$$\|u-v_h\|_{B_0} \le C\varepsilon_{2,1}(h).$$

We have already seen that the eigenvalue estimates in Steps A.1, A.3,... can be based on Lemma 2.3. Proceeding as we have here, we see that all of the eigenvalue estimates (3.2) can be based on Lemma 2.3.

4. Numerical Computations.

In the previous sections we have analyzed the errors in the Galerkin approximation of an eigenvalue problem, concentrating especially on the case of multiple eigenvalues. In this section we consider a finite element-Galerkin method for the approximation of a model, one-dimensional problem with multiple eigenvalues, presenting numerical results and their analysis in terms of the results of Section 3.

Consider the eigenvalue problem

$$(4.1)\qquad \begin{cases} -\left(\dfrac{1}{\varphi'(x)}\, u'(x)\right)' = \lambda\, \varphi'(x) u, \quad x \in I = (-\pi,\pi), \\ u(-\pi) = u(\pi), \\ \left(\dfrac{1}{\varphi'}\, u'\right)(-\pi) = \left(\dfrac{1}{\varphi'}\, u'\right)(\pi), \end{cases}$$

where

$$\varphi(x) = \pi^{-\alpha}|x|^{1+\alpha}\operatorname{sgn} x, \quad 0 < \alpha < 1.$$

It is easy to check that the eigenvalues and eigenfunctions are as shown in Table 4.1.

Table 4.1

Eigenvalues and Eigenfunctions of the Eigenvalue Problem (4.1)

i	λ_i	u_i
0	0.0	1
1	1.0	$\cos \varphi(x)$
2	1.0	$\sin \varphi(x)$
3	4.0	$\cos 2\varphi(x)$
4	4.0	$\sin 2\varphi(x)$
⋮	⋮	⋮

1,2,3,4, for $\alpha = .4$. These errors are plotted in Figure 4.1 in log-log scale. We clearly see the different rates of convergence, specifically seeing the rates h^2 and $h^{1+\alpha} = h^{1.4}$ for the errors in $\lambda_{h,i}$, for $i = 1,3$ and $i = 2,4$, respectively, as suggested by (4.2) and (4.3). It should be noted that the estimates presented in Theorem 3.1 are of an asymptotic nature in that they provide information only for small h (or large n), i.e., for h (or n) in the asymptotic range. From Figure 4.1 we see that for $\alpha = .4$ we are in the asymptotic range quite quickly, say for $n \geq 16$.

We computed $u_{h,1}$ and $u_{h,2}$, the approximate eigenfunctions corresponding to $\lambda_{h,1}$ and $\lambda_{h,2}$, respectively, normalized by $\|\cdot\|_D = 1$. The results of Section 3 suggest that $u_{h,1}$ should be close to $C \cos \varphi(x)$ and $u_{h,2}$ close to $C \sin \varphi(x)$ (cf. (4.2) and (4.3)), where C is such that $C \sin \varphi(x)$ and $C \cos \varphi(x)$ are normalized by $\|\cdot\|_D = 1$, i.e., $C = \pi^{-1/2}$. To illustrate this point we computed $C_1^{(i)}$ and $C_1^{(i)}$, $i = 1,2$, so that

$$(4.4)\quad K(i) = \begin{cases} \|u_{i,h} - C_1^{(i)} \cos \varphi(x) - C_2^{(i)} \sin \varphi(x)\|_{B_0}, & i = 1,2 \\ \|u_{i,h} - C_1^{(i)} \cos 2\varphi(x) - C_2^{(i)} \sin 2\varphi(x)\|_{B_0}, & i = 3,4 \end{cases}$$

is minimal. We would expect that

$$(4.5)\qquad C_1^{(2)},\ C_1^{(4)},\ C_2^{(1)},\ C_2^{(3)} \approx 0$$

and

$$(4.6)\qquad C_1^{(1)} = C_2^{(2)} = C_1^{(3)} = C_2^{(4)} \approx C = .564189583\ldots .$$

Table 4.2 shows some of the results for $\alpha = .4$. We see clearly

the results predicted in (4.5) and (4.6). The increase in $C_1^{(2)}$, $C_1^{(4)}$, $C_2^{(1)}$, and $C_2^{(3)}$ with increasing n is due to the eigenvalue solver we used. Table 4.2 also shows that $K(1) < K(2)$ and $K(3) < K(4)$, as we would expect.

The last column in Table 4.2 and Figure 4.1 show that the ratios

$$\frac{\lambda_{h,i+1} - \lambda_{i+1}}{\lambda_{h,i} - \lambda_i}, \quad i = 1,3,$$

increase as $h \longrightarrow 0$. This shows that in the whole h-range we considered, the approximate eigenvalues converging to a multiple eigenvalue are well separated.

Table 4.2

Numerical Solution of the Eigenvalue Problem (4.1) for α = .4

n	i	$\lambda_{h,i}$	$K(i)$		$C_1^{(i)}$		$C_2^{(i)}$		$\dfrac{\lambda_{h,i+1}-\lambda_{i+1}}{\lambda_{h,i}-\lambda_i}$
8	1	1.0716754	.2704	0	.5637791	0	-.1124891	-16	1.5562955
	2	1.1115481	.3423	0	-.4151973	-13	.5636998	0	
	3	5.0394692	.1075	+1	.5558919	0	.1317809	-12	1.1943249
	4	5.2414639	.1191	+1	.5022638	-13	.5516234	0	
16	1	1.0175850	.1329	0	.5641633	0	.1596754	-12	2.0041570
	2	1.0352431	.1881	0	-.8916589	-12	.5641519	0	
	3	4.2691915	.5259	0	.5636643	0	.1124328	-13	1.2575063
	4	4.3385100	.5869	0	-.2689727	-12	.5637697	0	
32	1	1.0043740	.6618	-1	.5641879	0	.6411454	-11	2.6003837
	2	1.0113741	.1067	0	.1323421	-10	.5641830	0	
	3	4.0666055	.2589	0	.5641561	0	.1970954	-10	1.4067517
	4	4.0936974	.3067	0	-.7375504	-10	.5641613	0	
64	1	1.0010921	.3305	-1	.5641895	0	.7729760	-9	3.5190001
	2	1.0038431	.6202	-1	.8670648	-9	.5641883	0	
	3	4.0166006	.1289	0	.5641875	0	.3641341	-10	1.6437659
	4	4.0272875	.1653	0	.1415775	-8	.5641858	0	
128	1	1.0002729	.1651	-1	.5641895	0	.4535626	-7	4.9215830
	2	1.0013431	.3665	-1	.3251219	-7	.5641893	0	
	3	4.0041468	.6440	-1	.5641895	0	.4409247	-7	2.0107071
	4	4.0083380	.9135	-1	-.9[illegible]05611	-8	.5641890	0	
256	1	1.0000682	.8255	-2	.5641896	0	.8070959	-5	7.0542522
	2	1.0004811	.2193	-1	.7269570	-6	.5641895	0	
	3	4.0010365	.3217	-1	.5641896	0	.6435344	-6	2.5706705
	4	4.0026645	.5162	-1	-.2601000	-6	.5641895	0	

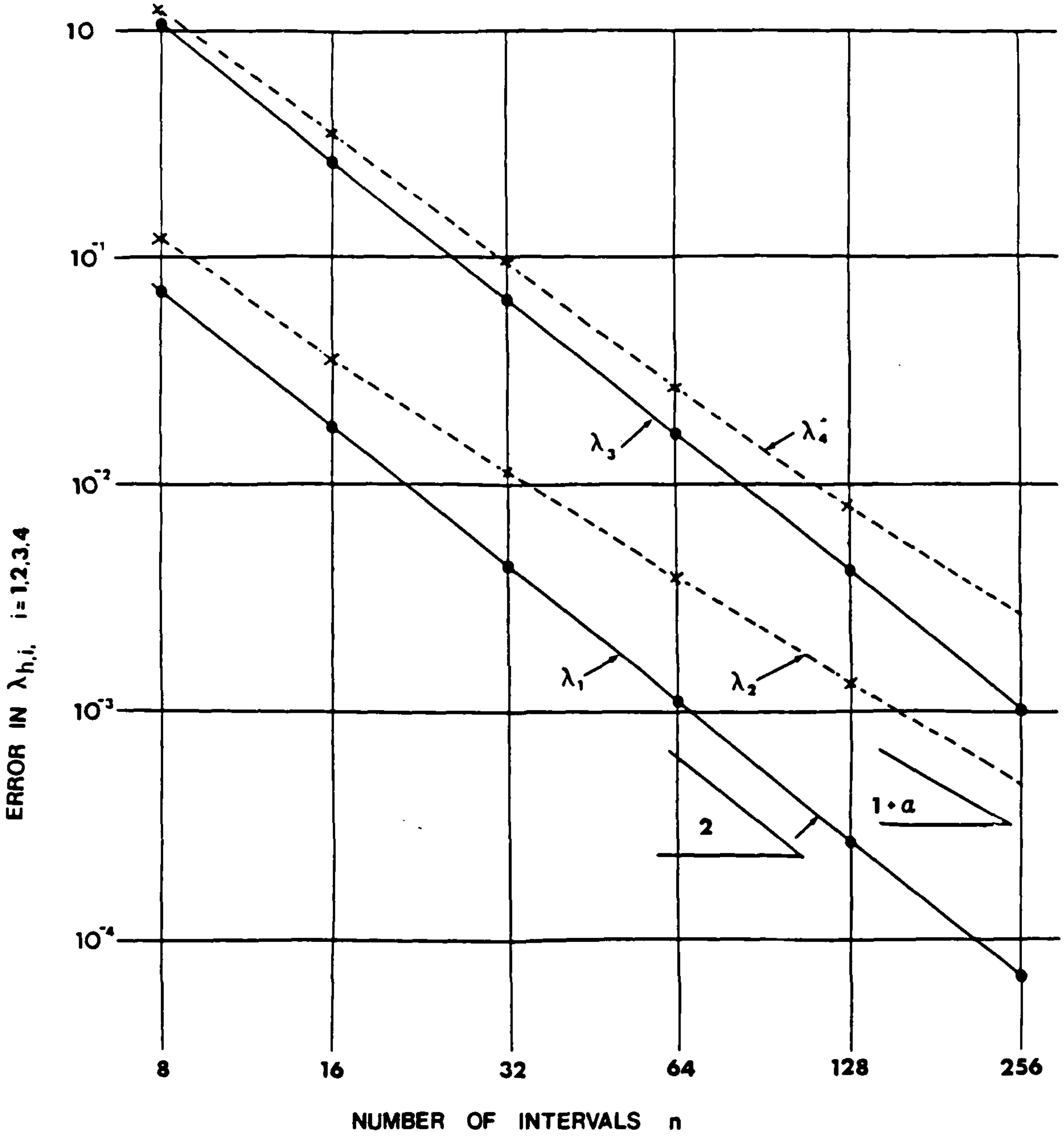

Figure 4.1

The Error in the Eigenvalues $\lambda_{h,1}, \lambda_{h,2}$ and $\lambda_{h,3}, \lambda_{h,4}$ for $\alpha = .4$ in Dependence on the Number of Intervals n

We next consider the case when $\alpha = .01$. Table 4.3 presents the same results for $\alpha = .01$ as Table 4.2 does for $\alpha = .4$. Figure 4.2 shows the graph of

$$\log \frac{\lambda_{h,i+1} - \lambda_{i+1}}{\lambda_{h,i} - \lambda_i}, \quad i = 1,3.$$

as a function of the number of intervals n in a semi-logarithmic scale. The computed values are indicated by o's and x's. The graphs are formed by interpolation (solid lines) and extrapolation (dotted lines). We note three related phenomena that did not occur with $\alpha = .4$. For small n the approximate eigenfunction associated with $\lambda_{h,1}$ is $u_{h,1} \approx \pi^{-1/2} \sin \varphi(x)$, in contrast to $u_{h,1} \approx \pi^{-1/2} \cos \varphi(x)$ when $\alpha = .4$. We remark that $\pi^{-1/2} \cos \varphi(x)$ is more easily approximated by S_h than is $\pi^{-1/2} \sin \varphi(x)$ for all $0 < \alpha < 1$. This anomaly is present for $n \le 64$ and for $n \ge 128$ we get results which are in agreement with the (asymptotic) results in Section 3. For $\lambda_{h,3}$ and $\lambda_{h,4}$ we have to take $n \ge 256$ to get results which agree with the asymptotic theory.

For $\alpha = .01$ we see that $K(2) < K(1)$ for small $n (n \le 64)$ and $K(2) > K(1)$ for large n and $K(4) < K(3)$ for small $n (n \le 128)$ and $K(4) > K(3)$ for large n. Recall that $K(2) > K(1)$ and $K(4) > K(3)$ for all n when $\alpha = .4$.

Finally we note that when $\alpha = .01$ the ratio

$$\frac{\lambda_{h,i+1} - \lambda_{i+1}}{\lambda_{h,i} - \lambda_i}, \quad i = 1,3,$$

first decreases as n increases, then for some n the two eigenvalue errors become equal, and then the ratio increases again.

This is in contrast to the case for $\alpha = .4$, in which the ratio increased over the whole range of n values. We further note that the value $\bar{n}$ for which the eigenvalue errors are equal — $\bar{n} \approx 70$ for $i = 1$ and $\bar{n} \approx 160$ for $i = 2$ (see Figure 4.2) — marks a transition in each of these situations from $u_{h,1} \approx \pi^{-1/2} \sin \varphi(x)$ to $u_{h,1} \approx \pi^{-1/2} \cos \varphi(x)$ and $u_{h,3} \approx \pi^{-1/2} \sin 2\varphi(x)$ to $u_{h,3} \approx \pi^{-1/2} \cos 2\varphi(x)$, from $K(2) < K(1)$ and $K(4) < K(3)$ to $K(2) > K(1)$ and $K(4) > K(3)$, and from $\frac{\lambda_{h,i+1} - \lambda_{i+1}}{\lambda_{h,i} - \lambda_i}$, $i = 1,3$, decreasing to increasing.

We have thus seen that for $\alpha = .4$ the numerical results are in concert with the (asymptotic) results in Section 3 for the whole range of n considered, while for $\alpha = .01$ they are in disagreement for small n, but are in agreement for large n. We now make an observation that further illuminates these two phases of error behavior — the pre-asymptotic and the asymptotic. Toward this end we note that if (λ_1, u_1), with $\|u_1\|_D = 1$, and $(\lambda_{h,1}, u_{h,1})$, with $\|u_{h,1}\|_D = 1$, are first eigenpairs of (2.5) and (2.11), respectively, then

$$
\begin{aligned}
0 \le \lambda_{h,1} - \lambda_1 &= \|u_{h,1} - u_1\|^2_{B_0} - \lambda_1 \|u_{h,1} - u_1\|^2_D \\
&= \inf_{\substack{v \in S_h \\ \|v\|_D = 1}} [\|v - u_1\|^2_{B_0} - \lambda_1 \|v - u_1\|^2_D]. \qquad (4.7)
\end{aligned}
$$

If λ_1 is a multiple eigenvalue, then the u_1 in (4.7) can be any corresponding eigenvector with $\|u_1\|_D = 1$. (Note that we are here assuming u_1 and $u_{h,1}$ have $\|\cdot\|_D$-length equal 1, whereas in (2.6) and (2.12) they are assumed to have $\|\cdot\|_{B_0}$-length equal

to 1.) The first inequality in (4.7) follows from the minimum principle (2.8*) and has already been stated in (2.13). The first equality in (4.7) follows immediately from Lemma 2.3 with $(\lambda,u) = (\lambda_1,u_1)$, $w = u_{h,1}$, and $\tilde{\lambda} = B_0(u_{h,1},u_{h,1}) = \lambda_{h,1}$. If $\psi \in S_h$ with $\|\psi\|_D = 1$, then from the minimum principle (2.8*),

$$\lambda_{h,1} - \lambda_1 \le B_0(\psi,\psi) - \lambda_1. \tag{4.8}$$

Again from Lemma 2.3, this time with with $(\lambda,u) = (\lambda_1,u_1)$, $w = \psi$, and $\tilde{\lambda} = B_0(\psi,\psi)$, we have

$$B_0(\psi,\psi) - \lambda_1 = \|\psi-u_1\|^2_{B_0} - \lambda_1\|\psi-u_1\|^2_D. \tag{4.9}$$

The second equality in (4.7) follows from (4.8) and (4.9). It is clear from the above discussion that u_1 can be any eigenvector corresponding to λ_1.

From (4.7) we have

$$\lambda_{h,1} - \lambda_1 \le \|\psi-u_1\|^2_{B_0} - \lambda_1\|\psi-u_1\|^2_D, \ \forall\, \psi \in S_h \ \text{ with } \|\psi\|_D = 1. \tag{4.10}$$

If ψ is $\|\cdot\|_{B_0}$-close to u_1, to be more precise, if ψ is taken to be the B_0-projection of u_1 onto S_h, then the second term as the right side of (4.10) is negligible with respect to the first term. This follows from the compactness assumption made on $\|\cdot\|_D$ in Section 2. On the other hand, if $\|u_1-\psi\|_{B_0}$ is not small, $\lambda_{h,1} - \lambda_1$ may still be small because of cancellation between the two terms on the right side of (4.10). Regarding the case $\alpha =$ .01, this explains why for h large (the pre-asymptotic phase), we can have $u_{h,1} \approx \pi^{-1/2} \sin \varphi(x)$ and $K(1) > K(2)$, and yet have $\lambda_{h,1}$, the approximate eigenvalue associated with $u_{h,1}$,

closer to λ_1 than is $\lambda_{h,2}$, the approximate eigenvalue associated with $u_{h,2} \approx \pi^{-1/2} \cos \varphi(x)$, while for h small (the asymptotic phase), we have $u_{h,1} \approx \pi^{-1/2} \cos \varphi(x)$, $K(1) < K(2)$, and $\lambda_{h,1}$ closer to λ_1 than is $\lambda_{h,2}$, showing that the eigenvalue error, $\lambda_{h,i} - \lambda_i$, is governed by $\inf\limits_{\chi \in S_h} \|\chi - u_1\|^2_{B_0}$.

This situation is very similar to the situation with n fixed and α varying, as can be seen from Table 4.4 where computations for the case $n = 4$ are shown. We see that the characteristics observed in Table 4.3 regarding dependence on n are present in Table 4.4 regarding dependence on α. Namely, the abrupt switch in the values of $C_1^{(i)}$, $C_2^{(i)}$, the abrupt switch from $K(2) < K(1)$ and $K(4) < K(3)$ to $K(2) > K(1)$ and $K(4) > K(3)$, and the abrupt switch from decreasing to increasing ratio of errors near the parameter value corresponding to $\lambda_{h,1} = \lambda_{h,2}$. We mention this situation — α varying and n fixed — since it is easier to understand in terms of perturbation theory (cf. Kato [5]) than is our original situation — n varying and α fixed.

Table 4.3

Numerical Solution of the Eigenvalue Problem (4.1) for ϵ = .01

n	i	$\lambda_{h,i}$	K(i)	$C_1^{(i)}$	$C_2^{(i)}$	$\dfrac{\lambda_{h,i+1}-\lambda_{i+1}}{\lambda_{h,i}-\lambda_i}$
8	1	1.0520268	.2338 0	.8181940 -11	.5634386 0	1.0171143
	2	1.0529172	.2268 0	.5645965 0	-.2916448 -11	
	3	4.8576239	.9593 0	-.9346720 -13	.5597529 0	1.0164293
	4	4.8717141	.9615 0	.5604533 0	.1167277 -11	
16	1	1.0128661	.1223 0	.8717399 -10	.5635957 0	1.0111689
	2	1.0130098	.1052 0	.5647369 0	-.8480131 -9	
	3	4.2088367	.4650 0	.2507177 -10	.5636658 0	1.0087030
	4	4.2106542	.4577 0	.5642694 0	-.3101833 -10	
32	1	1.0032139	.7274 -1	-.9345818 -9	.5636031 0	1.0068764
	2	1.0032360	.3568 -1	.5647430 0	.1273043 -7	
	3	4.0515675	.2384 0	.3745461 -9	.5638178 0	1.0057284
	4	4.0518629	.2205 0	.5644172 0	-.4115544 -9	
64	1	1.0008063	.5369 -1	-.1311961 -5	.5636032 0	1.0017363
	2	1.0008077	.3398 -1	.5647430 0	.2462939 -7	
	3	4.0128623	.1343 0	.2743681 -7	.5638240 0	1.0035997
	4	4.0129086	.9792 -1	.5644235 0	.3196172 -8	
128	1	1.0002018	.4196 -1	.5647430 0	.3356056 -5	1.0064420
	2	1.0002031	.4775 -1	.7414162 -6	.5636032 0	
	3	4.0032196	.9166 -1	.2379072 -6	.5638239 0	1.0010560
	4	4.0032230	.9745 -2	.5644235 0	.1197135 -5	
256	1	1.0000504	.4372 -1	.5647429 0	.1061527 -4	1.0216254
	2	1.0000515	.4614 -1	-.1553659 -4	.5636031 0	
	3	4.0008054	.5011 -1	.5644234 0	-.2123278 -4	1.0001040
	4	4.0008079	.7741 -1	.1165012 -5	.5638238 0	

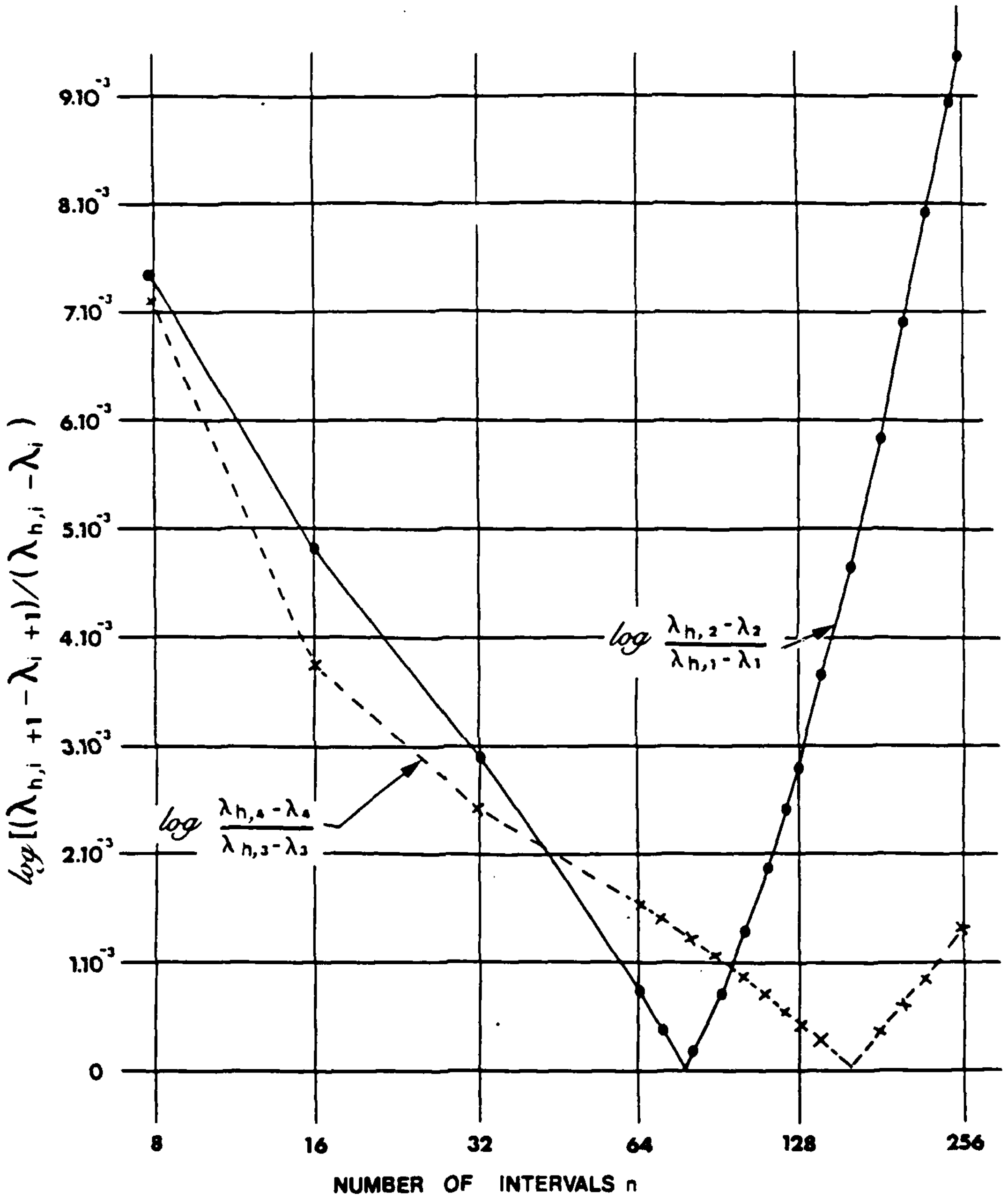

Figure 4.2

The Graphs of $\log \frac{\lambda_{h,2}-\lambda_2}{\lambda_{h,1}-\lambda_2}$ and $\log \frac{\lambda_{h,4}-\lambda_4}{\lambda_{h,3}-\lambda_3}$ for $\alpha = .01$ in Dependence on the Number of Intervals n.

Table 4.4

Numerical Solution of the Eigenvalue Problem (4.1) for $n = 4$

and for Various σ

σ	i	$\lambda_{h,i}$	K(i)		$C_1^{(i)}$		$C_2^{(i)}$		$\frac{\lambda_{h,2}^{-1}\lambda_2}{\lambda_{h,1}^{-1}\lambda_1}$
.1	1	1.2030785	.4689	0	-.2438763	-13	.5594106	0	1.2293000
	2	1.2496444	.5161	0	.5594467	0	.000		
.2	1	1.2238488	.5017	0	-.8324194	-14	.5562571	0	1.1969235
	2	1.2679299	.5366	0	.5585072	0	.000		
.25	1	1.2474367	.5322	0	-.8324194	-14	.5562571	0	1.0967540
	2	1.2713772	.5414	0	.5585072	0	.000		
.275	1	1.2628455	.5506	0	-.1979358	-12	.5526908	0	1.0342787
	2	1.2718555	.5426	0	.5577904	0	.000		
.28750	1	1.2715137	.5605	0	-.7895498	-11	.5520158	0	1.0011045
	2	1.2718136	.5428	0	.55768520	0	.5301589	-11	
.28779	1	1.2717254	.5607	0	.2170355	-10	.5519997	0	1.0003132
	2	1.2718105	.5429	0	.5576828	0	-.2773092	-10	
.28794	1	1.2718089	.5429	0	.5576816	0	.1753561	-10	1.0000809
	2	1.2718309	.5608	0	.1125467	-10	.5519917	0	
.28809	1	1.2718072	.5429	0	.5576804	0	.1631249	-11	1.0004794
	2	1.2719375	.5610	0	-.2645923	-11	.5519836	-6	
.28868	1	1.2718005	.5429	0	.5576757	0	.2249782	-16	[illegible]
	2	1.2723627	.5614	0	.3464439	-12	.5519513	0	
.29	1	1.2717839	.5429	0	.5576649	0	.2249782	-16	[illegible]
	2	1.2733271	.5625	0	.8906887	-13	.5518783	0	
.29735	1	1.2717263	.5429	0	.5576289	0	.000	0	[illegible]
	2	1.2760979	.5661	0	-.2000056	-13	.5515253	0	
.30	1	1.2715965	.5430	0	.5575862	0	.000	0	[illegible]
	2	1.2808526	.5790	0	-.1073461	-13	.5513203	0	
.40	1	1.2646804	.5382	0	.5570759	0	.2249782	-16	1.4473052
	2	1.3830892	.6704	0	.1185635	-13	.5451205	0	
.60	1	1.2364746	.5093	0	.5576799	0	.000	0	[illegible]
	2	1.8239095	.9695	0	.2249822	-16	.5304168	0	

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